Planar sets meeting each line in a set of measure 1

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joint work with Márton Elekes and Zoltán Vidnyánszky

The problem and its variants

Question (Existence of Steinhaus sets)

For a given $A \subseteq \mathbb{R}^2$ does there exist $H \subseteq \mathbb{R}^2$ such that for each rigid motion $\rho \in Iso(\mathbb{R}^2)$

$$|\rho(A) \cap H| = 1?$$

E.g. if one chooses $A = \mathbb{Z} \times \{0\}$, or $\mathbb{Z} \times \mathbb{Z}$?

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Question (Borel 2-point set)

Does there exists $H \subseteq \mathbb{R}^2$ Borel such that for each line $l \subseteq \mathbb{R}^2$

$$|l \cap H| = 2 \ ?$$

Facts

Theorem

(CH) There exists a set $H \subseteq \mathbb{R}^2$ such that for each line $l \subseteq \mathbb{R}^2$ ($l \cap H$ is λ^1-measurable, and)

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Proof. Let $\{l_\alpha : \alpha < \omega_1\}$ be the enumeration of the lines in the plane.

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For $\lambda^1(H \cap l_\alpha)$ note that

$$H_\alpha \subseteq H \cap l_\alpha$$

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$$H_\alpha \subseteq H \cap l_\alpha \subseteq H_\alpha \cup \left(\bigcup_{\beta<\alpha} (l_\alpha \cap l_\beta) \right).$$

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Remark

We only needed that each set of size less than continuum is null, i.e. $\mathrm{non}(\mathcal{N}) = \mathfrak{c}$.

Known results

Theorem

- *(Komjáth, '92) There exists* $H \subseteq \mathbb{R}^2$ *such that for each rigid motion* $\rho \in Iso(\mathbb{R}^n)$
$$|\rho(\mathbb{Z} \times \{0\}) \cap H| = 1.$$
- *(Jackson-Mauldin, 2002) There exists* $H \subseteq \mathbb{R}^2$*, for which for every* $\rho \in Iso(\mathbb{R}^n)$
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Theorem (Kolountzakis, Papadimitrakis, 2016.)

There is no λ^2*-measurable* $H \subseteq \mathbb{R}^2$, *for which*

$$\lambda^2\text{-a.e. } x \in \mathbb{R}^2 \quad \text{a.e. } l \ni x : \quad \lambda^1(H \cap l) = 1.$$

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Consider $H \subseteq \mathbb{R}^2 \times \{0\}$, $\chi_H : \mathbb{R}^3 \to \{0, 1\}$.

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Define $h : \mathbb{R}^3 \to [0, \infty]$, $h(z) = \int_{\mathbb{R}^2} \chi_H(w) \cdot \frac{1}{|w-z|} d\lambda^2(w)$.

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Then for λ^2-a.e. $x \in \mathbb{R}^2 \times \{0\}$

$$h(x) = \int_{\theta=0}^{\pi} \int_{r=-\infty}^{\infty} \chi_H(x+r(\sin(\theta),\cos(\theta),0)) \cdot \frac{1}{|x+r(\sin(\theta),\cos(\theta),0)-x|} \cdot |r| \, dr d\theta =$$

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This implies (integrating with Poisson-kernel) that $h|_{\mathbb{R}^2 \times \{0\}} \equiv \pi$ determines $h|_{\mathbb{R}^2 \times [0,\infty)}$, hence $h|_{\mathbb{R}^2 \times [0,\infty)} \equiv \pi$.

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Consistency of the nonexistence

Theorem (M. Elekes, M. P., Z. Vidnyánszky)

It is consistent with ZFC that there is no set $H \subseteq \mathbb{R}^2$ such that for a dense $D \subseteq \mathbb{R}^2$

$$\textit{if } x \in D, \textit{ then for a.e. } l \ni x: \quad \lambda^1(l \cap H) = 1.$$

Tools

Definition

$$\begin{aligned} \text{shr}(\mathcal{N}) = & \quad \min\{\kappa : \ \forall A \subseteq \mathbb{R}, A \notin \mathcal{N} \ \exists B \subseteq A \ \textit{ such that } B \notin \mathcal{N}, \ |B| \leq \kappa\}, \\ \text{cov}(\mathcal{N}) = & \quad \min\{\kappa : \ \exists (N_\alpha)_{\alpha<\kappa} \ \forall \alpha N_\alpha \in \mathcal{N} \wedge \bigcup_{\alpha<\kappa} N_\alpha = \mathbb{R}\} \end{aligned}$$

(where $\mathcal{N}$ denotes the null ideal in $\mathbb{R}$).

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Theorem (Folklore)

($\mathrm{shr}(\mathcal{N}) < \mathrm{cov}(\mathcal{N})$) *Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ has λ^1-measurable sections (i.e. for each a $f_a : \mathbb{R} \to \mathbb{R}$ ($f_a(y) = f(a, y)$) is λ^1-measurable, and for each b $f^b(x) = f(x, b)$ is λ^1-measurable.)*

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Then there exists $f' : \mathbb{R}^2 \to \mathbb{R}$ Borel such that for almost every $a \in \mathbb{R}$ f_a and f'_a are almost equal, and for a.e. b, f^b and $(f')^b$ are almost equal.

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Then there exists $f' : \mathbb{R}^2 \to \mathbb{R}$ Borel such that for almost every $a \in \mathbb{R}$ f_a and f'_a are almost equal, and for a.e. b, f^b and $(f')^b$ are almost equal.
In particular, if $f \geq 0$ then $y \mapsto \int_{x=-\infty}^{\infty} f(x, y)\ dx$, $x \mapsto \int_{y=-\infty}^{\infty} f(x, y)\ dy$ are λ^1-measurable, and

$$\int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} f(x, y)\ dxdy = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} f(x, y)\ dydx.$$

The idea

Theorem (M. Elekes, M. P., Z. Vidnyánszky)

It is consistent with ZFC that there is no set $H \subseteq \mathbb{R}^2$ such that for a dense $D \subseteq \mathbb{R}^2$

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Assume that $\mathrm{shr}(\mathcal{N}) < \mathrm{cov}(\mathcal{N})$. Suppose that such H exists. (For simplicity assume that H has measurable vertical and horizontal sections.)

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Define $h : \mathbb{R}^3 \to [0, \infty]$ similarly

$$h(w) = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} \chi_H((x,y)) \cdot \frac{1}{|(x,y,0) - w|} dxdy.$$

h will have similar properties than the previous one, but we cannot use polar transformation (for $h|_D \equiv \pi$).

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h will have similar properties than the previous one, but we cannot use polar transformation (for $h|_D \equiv \pi$).
Instead, for a fixed $w \in D$ denoting $\chi_H((x,y)) \cdot \frac{1}{|(x,y,0)-w|}$ by $\mathbf{f_w(x,y)}$, and fixing an automorphism of the projective plane $\mathbf{g}$ which maps horizontal lines to horizontal lines, and vertical lines to lines through w, and

$$\varphi : [0, \pi] \times \mathbb{R} \to \mathbb{R}^2, \quad \varphi(\theta, r) = w + r(\sin(\theta), \cos(\theta)).$$

The key lemmas

$$h(w)=\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty}\chi_H((x,y))\cdot\frac{1}{|(x,y,0)-w|}dxdy.$$

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$$\int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} f_w(x,y) dxdy = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} f_w(g(x,y)) \cdot |\det g'(x,y)| dxdy$$

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The key lemmas

$$h(w) = \int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty} \chi_H((x,y)) \cdot \frac{1}{|(x,y,0)-w|} dxdy.$$

For a fixed $w \in D$ denoting $\chi_H((x,y)) \cdot \frac{1}{|(x,y,0)-w|}$ by $\mathbf{f_w(x,y)}$, and fixing an automorphism of the projective plane $\mathbf{g}$ which maps horizontal lines to horizontal lines, and vertical lines to lines through w, and

$$\varphi : [0,\pi] \times \mathbb{R} \to \mathbb{R}^2, \quad \varphi(\theta, r) = w + r(\sin(\theta), \cos(\theta)).$$

Lemma

$\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty} f_w(x,y)dxdy = \int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty} f_w(g(x,y)) \cdot |\det g'(x,y)|dxdy$ *(where* $\det g'(x,y) = \partial_x g_1(x,y) \cdot \partial_y g_2(x,y) = \partial_x g_1(x,y) \cdot \partial_y g_2(0,y)$*).*

Lemma

$\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} f_w(g(x,y)) \cdot |\det g'(x,y)|dydx =$
$\int_{\theta=0}^{\pi}\int_{r=-\infty}^{\infty} f_w(\varphi(\theta,r)) \cdot |\det \varphi'(\theta,r)|drd\theta = \pi.$

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$$h(w) = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} \chi_H((x,y)) \cdot \frac{1}{|(x,y,0)-w|} dxdy.$$

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Lemma

$\int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} f_w(x,y)dxdy = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} f_w(g(x,y)) \cdot |\det g'(x,y)| dxdy$
(where $\det g'(x,y) = \partial_x g_1(x,y) \cdot \partial_y g_2(x,y) = \partial_x g_1(x,y) \cdot \partial_y g_2(0,y)$*).*

Lemma

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with introducing $f_0(x,y) = f_w(g(x,y)) \cdot |\det g'(x,y)|$,
$f_1(r,\theta) = f_w(\varphi(\theta,r)) \cdot |\det \varphi'(\theta,r)|$ and the transformation $\varphi^{-1} \circ g$

Thank you for your attention!

References

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